In class notes - quick sort, pt 2

CS 14 - Data Structures

April 15, 2013

When analyzing the runtime of quicksort, we found that in a worst case scenario, the run time obeys the recurrence relation:

$$T(n) = T(n-1) + T(1) + n$$

Applying our base case of $T(1) = 1$, we can simplify to:

$$T(n) = T(n-1) + n$$

Now, I'm going to calculate the following "intermediate equations":

$$\begin{aligned} T(n-1) &= T(n-2) + n - 1 \\ T(n-2) &= T(n-3) + n - 2 \\ T(n-3) &= T(n-4) + n - 3 \end{aligned}$$

We can use these equations to solve $T(n)$ by substituting:

$$\begin{aligned} T(n) &= T(n-1) + n \\ &= [T(n-2) + n - 1] + n \qquad = T(n-2) + 2n - 1 \\ &= [T(n-3) + n - 2] + 2n - 1 = T(n-3) + 3n - (1+2) \\ &= [T(n-4) + n - 3] + 3n - 3 = T(n-4) + 4n - (1+2+3) \\ &= T(n-k) + kn - \sum_{i=0}^{k} i \\ &= T(n-k) + kn - \frac{k(k-1)}{2} \\ &= T(n-k) + kn - \frac{k^2 - k}{2} \end{aligned}$$

Next, we need to apply the base case:

$$\begin{aligned} n - k &= 1 \\ k &= n - 1 \end{aligned}$$

And substitute it back in:

$$\begin{aligned} T(n) &= 1 + (n-1)n - \frac{(n-1)^2 - (n-1)}{2} \\ &= 1 + n^2 - n - \frac{\left(n^2 - 2n + 1\right) - (n-1)}{2} \\ &= n^2 - \frac{n^2}{2} - n - \frac{-3n}{2} + 1 - \frac{1}{2} \\ &= \Theta\left(n^2\right) \end{aligned}$$

When analyzing the runtime of quicksort, we also found that the average case run time obeys the recurrence relation:

$$T(n) = 2T\left(\frac{n}{2}\right) + n$$

Now, I'm going to calculate the following "intermediate equations":

$$\begin{aligned}
T\left(\frac{n}{2}\right) &= 2T\left(\frac{n}{4}\right) + \frac{n}{2} \\
T\left(\frac{n}{4}\right) &= 2T\left(\frac{n}{8}\right) + \frac{n}{4} \\
T\left(\frac{n}{8}\right) &= 2T\left(\frac{n}{16}\right) + \frac{n}{8}
\end{aligned}$$

We can use these equations to solve $T(n)$ by substituting:

$$\begin{aligned}
T(n) &= 2T\left(\frac{n}{2}\right) + n \\
&= 2[2T\left(\frac{n}{4}\right) + \frac{n}{2}] + n = 4T\left(\frac{n}{4}\right) + 2n \\
&= 4[2T\left(\frac{n}{8}\right) + \frac{n}{4}] + 2n = 8T\left(\frac{n}{8}\right) + 3n \\
&= 8[2T\left(\frac{n}{16}\right) + \frac{n}{8}] + 3n = 16T\left(\frac{n}{16}\right) + 4n \\
&= 2^k T\left(\frac{n}{2^k}\right) + kn
\end{aligned}$$

Next, we need to apply the base case:

$$\begin{aligned}
\frac{n}{2^k} &= 1 \\
n &= 2^k \\
k &= \log_2 n
\end{aligned}$$

And substitute it back in:

$$\begin{aligned}
T(n) &= 2^{\log_2 n} T(1) + n \log_2 n \\
&= n + n \log_2 n \\
&= \Theta(n \log n)
\end{aligned}$$